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Recall:

Theorem 3.4 (Perturbation of Identity)
Let $(X, \|\cdot\|)$ be a Banach space and $\Phi : \overline{B_r(x_0)} \to X$ satisfies $\Phi(x_0) = y_0$. Suppose that Φ is of the form $I + \Psi$ where I is the identity map and Ψ satisfies

$$\|\Psi(x_2) - \Psi(x_1)\| \leq \gamma \|x_2 - x_1\|, \quad x_1, x_2 \in \overline{B_r(x_0)}, \gamma \in (0,1)$$

Then for $y \in \overline{B_R(y_0)}$, $R = (1-\gamma)r$, there is a unique $x \in \overline{B_r(x_0)}$ satisfying $\Phi(x) = y$.

Theorem 3.7 (Inverse Function Theorem)
Let $F : U \to \mathbb{R}^n$ be a C^1-map where U is open in $\mathbb{R}^n$ and $x_0 \in U$. Suppose that $DF(x_0)$ is invertible.

(a) There exists open sets V and W containing x_0 and $F(x_0)$ respectively such that the restriction of F on V is a bijection onto W with C^1-inverse

(b) The inverse is C^k when F is C^k, for any $1 \leq k \leq \infty$, in V

Theorem 3.9 (Implicit Function Theorem)
Consider a C^1-map $F : U \to \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose that $(x_0, y_0) \in U$ satisfies $F(x_0, y_0) = 0$ and $D_y F(x_0, y_0)$ is invertible in $\mathbb{R}^m$. There exists an open set $V_1 \times V_2$ in U containing (x_0, y_0) and a C^1-map $\varphi : V_1 \to V_2$, $\varphi(x_0) = y_0$, such that

$$F(x, \varphi(x)) = 0$$

The map φ belongs to C^k when F is C^k, for $1 \leq k \leq \infty$, in U. Moreover, assume further that DF_y is invertible in $V_1 \times V_2$. If $\psi : V_1 \to V_2$ is a C^1-map satisfying $F(x, \psi(x)) = 0$, then $\psi = \varphi$.

Remark: A map $F : \mathbb{R}^n \to \mathbb{R}^m$ is called a *diffeomorphism* if it (i) is smooth, (ii) is bijective and (iii) has a smooth inverse. A map F is called a *local diffeomorphism* around p if there exists an open neighborhood $U \subset \mathbb{R}^n$ of p and an open neighborhood $V \subset \mathbb{R}^m$ of $F(p)$ such that $F|_U : U \to V$ is a diffeomorphism.

Exercise 1

Source: Previous Homework Problem

Show that $\begin{cases} x + y^4 = 0 \\ y - x^2 = 0.015 \end{cases}$ is solvable near $(x, y) = 0 \in (\mathbb{R}^2, \|\cdot\|)$

Note that $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^2$.

Solution:
Define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi = I + \Psi$, where $\Psi(x, y) = (\Psi_1(x, y), \Psi_2(x, y)) = (y^4, -x^2)$. We can see that $\Phi(0, 0) = (0, 0)$.

Then we can apply Perturbation of Identity to Φ. The idea is to construct a $r > 0$ such that $\Psi|_{\overline{B_r(0)}} : (\overline{B_r(0)}, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ is a contraction. For any $(x_1, y_1), (x_2, y_2) \in \overline{B_r(0)}$, we have

$$\|\Psi(x_1, y_1) - \Psi(x_2, y_2)\| \leq M \|(x_1, y_1) - (x_2, y_2)\|$$

where

$$M := \sup_{(x,y) \in \overline{B_r(0)}} \left(\left(\frac{\partial \Psi_1}{\partial x}\right)^2 + \left(\frac{\partial \Psi_1}{\partial y}\right)^2 + \left(\frac{\partial \Psi_2}{\partial x}\right)^2 + \left(\frac{\partial \Psi_2}{\partial y}\right)^2 \right)^{\frac{1}{2}}$$

as in P.8 of lecture notes 3 by Prof K.S. Chou. We calculate M explicitly and obtain

$$\sup_{(x,y) \in \overline{B_r(0)}} \left(0^2 + 16y^6 + 4x^2 + 0\right)^{\frac{1}{2}} \leq 2r\sqrt{4r^4 + 1}$$

since $\|(x, y)\| \leq r \implies x, y \leq r$.

Choose $r = \frac{1}{4}$, then $M \leq 2\left(\frac{1}{4}\right)\sqrt{4\left(\frac{1}{4}\right)^4 + 1} = \frac{\sqrt{65}}{16} < 1$. Hence, we have show that

$$\Psi|_{\overline{B_r(0)}} : (\overline{B_r(0)}, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$$

is a contraction.

By Perturbation of Identity, $\Phi(x) = y$ is solvable for any $y \in \overline{B_R(0)}$, here, 0 is because $\Phi(0) = 0$, and $R = (1 - M)r = (1 - \frac{\sqrt{65}}{16})\frac{1}{4} = \frac{16 - \sqrt{65}}{64}$. In particular, $(0, 0.015) \in \overline{B_R(0)}$. Thus, the given system is solvable for $(x, y) \in \overline{B_{\frac{1}{4}}(0)}$.

■

Remark: The "M-trick" here is useful. Analysis has a lot of so-called "tricks", you can remember some of them as they come in handy when you need it.

Exercise 2

Source: Modified from Exercise 6.7.3 in Terence Tao's "Analysis II"[1].

This exercise serves as a Corollary of the Inverse Function Theorem.

Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 such that its Jacobian, $DF(x)$, is nonsingular for every $x \in \mathbb{R}^n$. Show that whenever U is open in $\mathbb{R}^n$, $F(U)$ is open in $\mathbb{R}^n$

Solution:
For any $p \in \mathbb{R}^n$, we are given that

$$\det DF(p) \neq 0$$

then the inverse function theorem tells us that there exists an open neighborhood V_p containing p and an open neighborhood W_p containing $F(p)$ such that $F|_{V_p} : V_p \to W_p$ is a diffeomorphism. In particular, a diffeomorphism is a homeomorphism, this means $F(V_p)$ is open.

Now that for any open set $U \subset \mathbb{R}^n$, at any point $x \in U$, we have a neighborhood V_x such that $F(V_x)$ is open from the above discussion. Since $U = \bigcup_{x \in U} V_x$, then

$$F(U) = \bigcup_{x \in U} F(V_x)$$

is open.

■

[1]You can download Tao's book (actually almost every Springer books) on Springer's website using your CUHK student credentials in here.

Exercise 3

Source: Midterm Question of MATH3043 at HKUST written by Prof Frederick Fong

The goal of this exercise is to show how the inverse function theorem can be applied.

Consider a C^1 function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\Sigma := f^{-1}(0)$ is nonempty and $\nabla f(p) \neq 0$ for any $p \in \Sigma$. Show that for any $p \in \Sigma$, there exists a bijective map $\varphi : U \to V$, where U is open in $\mathbb{R}^3$ containing p and V is another open set in $\mathbb{R}^3$ so that both φ and φ^{-1} are C^1, and that

$$\Sigma \cap U = \left\{ \varphi^{-1}(x, y, 0) : (x, y, 0) \in V \right\}$$

Solution:
Since $\nabla f(p) \neq 0$, we may assume, WLOG, that $\frac{\partial f}{\partial z}(p) \neq 0$.

Now we construct a map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\psi(x, y, z) := (x, y, f(x, y, z))$$

What this map does is to "straighten" a curved surface to a plane. For example, if ψ takes value from Σ, then $\psi(x, y, z) = (x, y, 0)$ which is essentially the plane $z = 0$ in $\mathbb{R}^3$.

Next, we want to find φ as stated in the question. Consider locally at p,

$$D\psi(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix}$$

the determinant is given by $\frac{\partial f}{\partial z}(p)$, which is assumed to be nonzero from the condition that $\nabla f(p) \neq 0$. So, the Jacobian is nonsingular, the inverse function theorem then tells us that there exists an open set U containing p and a open set V containing $\psi(p)$ such that

$$\varphi := \psi|_U : U \to V$$

is a bijection with C^1-inverse.

Lastly, we check $\Sigma \cap U = \{\varphi^{-1}(x, y, 0) : (x, y, 0) \in V\}$. For all $q \in U \cap \Sigma$, we have $f(q) = 0$. This implies $\psi(q) = (x, y, f(x, y, z)) = (x, y, 0) \in V$. Hence, we have the first inclusion.

Now take any $q \in \{\varphi^{-1}(x, y, 0) : (x, y, 0) \in V\}$, then $\varphi(q) = (x, y, 0)$, this shows that $f(q) = 0$ and hence the second inclusion is proved.

$\blacksquare$

Remark: One can also check Proposition 2 in P.61 of do Carmo's "Differential Geometry of Curves and Surfaces", which basically says the same thing with some terminologies from differential geometry. Interestingly, I found that "curves and surfaces" was indeed a topic in MATH3060 before, but is now replaced by the Picard-Lindelöf Theorem.

Exercise 4 (Optional)

This question is **beyond the scope of this course**. But this questions introduces how inverse function theorem is useful in handling geometric problems. If you are interested, you can take MATH4030, MATH5070 and MATH5061, or, if you want a quick introduction, you can read "Differentiable Manifolds and Riemannian Geometry" written by Prof Frederick Fong, you can find it here.

The goal of this question is to show that the unit cylinder,

$$M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$$

is diffeomorphic to the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ via the map $\Phi : M \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$\Phi(x, y, z) = e^z(x, y)$$

(a) Show that Φ is bijective.

(b) Find two parametrization of M. [The reason of finding two parametrizations is because one cannot cover the whole M.]

(c) Denote the two parametrizations by F_1 and F_2, then show that $\Phi \circ F_i$ is smooth and that its inverse $F_i^{-1} \circ \Phi^{-1}$ is smooth too, for $i = 1, 2$. [This is the part where Inverse Function Theorem is used].

Solution: See Example 2.23 from Prof Frederick Fong's notes.

■